Note on the reduction of Alperin's Conjecture

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1. In a recent paper [2], Gabriel Navarro and Pham Huu Tiep show that the so-called Alperin Weight Conjecture can be verified *via* the *Classification of the Finite Simple Groups*, provided any simple group fulfills a very precise list of conditions that they consider easier to check than ours, firstly stated in [3, Theorem 16.45] and significantly weakened in [4, Theorem 1.6]†.

2. Actually, in the introduction of [3] — from I29 to I37 — we consider the most precise Alperin's Conjecture concerning any block of a finite group, and introduce a refinement to this conjecture; but, only in [4] we really show that its verification can be reduced to check that the *same* refinement holds on the so-called *quasi-simple* groups. To carry out this checking obviously depends on admitting the *Classification of the Finite Simple Groups*, and our proof of the reduction itself uses the *solvability* of the *outer automorphism group* of a finite simple group, a known fact whose actual proof depends on this classification.

3. Our purpose here is to show to the interested reader†† that the results in [3] and the reduction arguments in [4] suggest a numerical statement — implying Alperin's Conjecture — which can be reduced again to check that the same holds on the quasi-simple groups and, this time, this statement on the quasi-simple groups follows from the list of conditions demanded in [2].

4. Let us be more explicit. Let p be a prime number, k an algebraically closed field of characteristic p, $\mathcal{O}$ a complete discrete valuation ring of characteristic zero admitting k as the *residue* field, and $\mathcal{K}$ the field of fractions of $\mathcal{O}$. Moreover, let $\hat{G}$ be a k^*-group of finite k^*-quotient G [3, 1.23], b a block of $\hat{G}$ [3, 1.25] and $\mathcal{G}_k(\hat{G}, b)$ the *scalar extension* from $\mathbb{Z}$ to $\mathcal{O}$ of the *Grothendieck group* of the category of finitely generated $k_*\hat{G}b$-modules [3, 14.3]. In [3, Chap. 14], choosing a maximal Brauer $(b, \hat{G})$-pair (P, e), the existence of a suitable k^*-$\mathfrak{Gr}$-valued functor $\widehat{\mathfrak{aut}}_{(\mathcal{F}_{(b,\hat{G})})^{\mathrm{nc}}}$ over some full subcategory $(\mathcal{F}_{(b,\hat{G})})^{\mathrm{nc}}$ of the *Frobenius P-category* $\mathcal{F}_{(b,\hat{G})}$ [3, 3.2] allows us to consider an inverse limit of Grothendieck groups — noted $\mathcal{G}_k(\mathcal{F}_{(b,\hat{G})}, \widehat{\mathfrak{aut}}_{(\mathcal{F}_{(b,\hat{G})})^{\mathrm{nc}}})$

† Gabriel Navarro and Pham Huu Tiep pointed out to us that, when submitting [2], they were not aware of our paper [4], only available in arXiv since April 2010.

†† Indeed, a first submission of [4] has been rejected since, according to the referee, "the results (...) are too specialized for Inventiones".

and called the *Grothendieck group of* $\mathcal{F}_{(b,\hat{G})}$ — such that Alperin's Conjecture is actually equivalent to the existence of an $\mathcal{O}$-module isomorphism [3, I32 and Corollary 14.32]

$$\mathcal{G}_k(\hat{G},b) \cong \mathcal{G}_k(\mathcal{F}_{(b,\hat{G})}, \widehat{\mathfrak{aut}}_{(\mathcal{F}_{(b,\hat{G})})^{\mathrm{nc}}}) \qquad 4.1.$$

5. Denote by $\mathrm{Out}_{k^*}(\hat{G})$ the group of *outer* k^*-automorphisms of $\hat{G}$ and by $\mathrm{Out}_{k^*}(\hat{G})_b$ the stabilizer of b in $\mathrm{Out}_{k^*}(\hat{G})$; it is clear that $\mathrm{Out}_{k^*}(\hat{G})_b$ acts on $\mathcal{G}_k(\hat{G},b)$, and in [3, 16.3 and 16.4] we show that this group still acts on $\mathcal{G}_k(\mathcal{F}_{(b,\hat{G})}, \widehat{\mathfrak{aut}}_{(\mathcal{F}_{(b,\hat{G})})^{\mathrm{nc}}})$. Denoting by

$$^{\mathcal{K}}\mathcal{G}_k(\hat{G},b) \quad \text{and} \quad {^{\mathcal{K}}\mathcal{G}_k}(\mathcal{F}_{(b,\hat{G})}, \widehat{\mathfrak{aut}}_{(\mathcal{F}_{(b,\hat{G})})^{\mathrm{nc}}}) \qquad 5.1$$

the respective *scalar extensions* of $\mathcal{G}_k(\hat{G},b)$ and $\mathcal{G}_k(\mathcal{F}_{(b,\hat{G})}, \widehat{\mathfrak{aut}}_{(\mathcal{F}_{(b,\hat{G})})^{\mathrm{nc}}})$ from $\mathcal{O}$ to $\mathcal{K}$, here we replace the statement (Q) in [4, 1.4] by the following statement

($^{\mathcal{K}}$Q) *For any* k^**-group* $\hat{G}$ *with finite* k^**-quotient* G *and any block* b *of* $\hat{G}$, *there is an* $\mathcal{K}\mathrm{Out}_{k^*}(\hat{G})_b$*-module isomorphism*

$$^{\mathcal{K}}\mathcal{G}_k(\hat{G},b) \cong {^{\mathcal{K}}\mathcal{G}_k}(\mathcal{F}_{(b,\hat{G})}, \widehat{\mathfrak{aut}}_{(\mathcal{F}_{(b,\hat{G})})^{\mathrm{nc}}}) \qquad 5.2$$

6. The big difference between the statements (Q) and ($^{\mathcal{K}}$Q) is that the second one is equivalent to the equality of the corresponding $\mathcal{K}$*-characters* of $\mathrm{Out}_{k^*}(\hat{G})_b$; moreover, since these $\mathcal{K}\mathrm{Out}_{k^*}(\hat{G})_b$-modules actually come from $\mathbb{Q}\mathrm{Out}_{k^*}(\hat{G})_b$-modules, isomorphism 5.2 is finally equivalent to the equalities

$$\mathrm{rank}_{\mathcal{O}}\big(\mathcal{G}_k(\hat{G},b)^C\big) = \mathrm{rank}_{\mathcal{O}}\big(\mathcal{G}_k(\mathcal{F}_{(b,\hat{G})}, \widehat{\mathfrak{aut}}_{(\mathcal{F}_{(b,\hat{G})})^{\mathrm{nc}}})^C\big) \qquad 6.1$$

for any cyclic subgroup C of $\mathrm{Out}_{k^*}(\hat{G})_b$. On the other hand, without any change, our proof of [4, Theorem 1.6] still proves the following result.

Theorem 7. *Assume that any block* $(c,\hat{H})$ *having a normal sub-block* $(d,\hat{S})$ *of positive defect such that the* k^**-quotient* S *of* $\hat{S}$ *is simple,* H/S *is a cyclic* p'*-group and* $C_H(S)=\{1\}$, *fulfills the following two conditions*

7.1 $\mathrm{Out}(S)$ *is solvable.*

7.2 *For any cyclic subgroup* C *of* $\mathrm{Out}_{k^*}(\hat{H})_c$ *we have*

$$\mathrm{rank}_{\mathcal{O}}\big(\mathcal{G}_k(\hat{H},c)^C\big) = \mathrm{rank}_{\mathcal{O}}\big(\mathcal{G}_k(\mathcal{F}_{(c,\hat{H})}, \widehat{\mathfrak{aut}}_{(\mathcal{F}_{(c,\hat{H})})^{\mathrm{nc}}})^C\big)\ .$$

Then, for any block $(b,\hat{G})$ *there is an* $\mathcal{K}\mathrm{Out}_{k^*}(\hat{G})_b$*-module isomorphism*

$$^{\mathcal{K}}\mathcal{G}_k(\hat{G},b) \cong {^{\mathcal{K}}\mathcal{G}_k}(\mathcal{F}_{(b,\hat{G})}, \widehat{\mathfrak{aut}}_{(\mathcal{F}_{(b,\hat{G})})^{\mathrm{nc}}}) \qquad 7.3.$$

8. Morover, with the same notation of the theorem, it is easily checked that the proof of [3, Corollary 14.32] actually also proves that

$$\begin{aligned}\mathrm{rank}_{\mathcal{O}}\big(\mathcal{G}_k(\mathcal{F}_{(c,\hat H)},\widehat{\mathfrak{aut}}_{(\mathcal{F}_{(c,\hat H)})^{\mathrm{nc}}})^C\big)\\ = \sum_{(\mathfrak{q},\Delta_n)}(-1)^n\,\mathrm{rank}_{\mathcal{O}}\Big(\mathcal{G}_k\big(\hat{\mathcal{F}}_{(c,\hat H)}(\mathfrak{q})\big)^{C_{\mathfrak{q}}}\Big)\end{aligned} \tag{8.1}$$

where $(\mathfrak{q},\Delta_n)$ runs over a set of representatives for the set of isomorphism classes of regular $\mathfrak{ch}^*(\mathcal{F}^{^{\mathrm{sc}}}_{(c,\hat H)})$-objects [3, 45.2] and, for such a $(\mathfrak{q},\Delta_n)$, $C_{\mathfrak{q}}$ denotes the "stabilizer" of $(\mathfrak{q},\Delta_n)$ in C (see [3, 15.33] for a similar notation); indeed, in that proof, the sequence of C-fixed elements in the exact sequence [3, 14.32.4] remains exact, since we are working over $\mathcal{K}$, and then equality 8.1 follows easily.

9. At this point, the old argument of Reinhard Knörr and Geoffrey Robinson in [1], suitably adapted, shows that, when proving statement $({}^{\mathcal{K}}\mathrm{Q})$ arguing by induction on $|G|$, we may assume that

$$\begin{aligned}\sum_{(\mathfrak{q},\Delta_n)}(-1)^n\,\mathrm{rank}_{\mathcal{O}}\Big(\mathcal{G}_k\big(\hat{\mathcal{F}}_{(c,\hat H)}(\mathfrak{q})\big)^{C_{\mathfrak{q}}}\Big)\\ = \sum_{(Q,f)}\mathrm{rank}_{\mathcal{O}}\Big(\mathcal{G}_k\big(\hat{\mathcal{F}}_{(c,\hat H)}(Q,f),\bar b_f\big)^{C_f}\Big)\end{aligned} \tag{9.1}$$

where (Q,f) runs over a set of representatives for the set of H-conjugacy classes of selfcentralizing Brauer $(c,\hat H)$-pairs [3, 1.16 and Corollary 7.3] and, for such a selfcentralizing Brauer $(c,\hat H)$-pair (Q,f), we denote by $\bar b_f$ the sum of blocks of defect zero of $\hat{\mathcal{F}}_{(c,\hat H)}(Q,f)/\mathcal{F}_Q(Q)$, and by C_f the "stabilizer" of (Q,f) in C.

10. That is to say, according to equalities 8.1 and 9.1, when proving Theorem 7 arguing by induction on $|G|$ we may replace condition 7.2 by the alternative condition:

10.1 *For any cyclic subgroup C of $\mathrm{Out}_{k^*}(\hat H)_c$ we have*

$$\mathrm{rank}_{\mathcal{O}}\big(\mathcal{G}_k(\hat H,c)^C\big) = \sum_{(Q,f)}\mathrm{rank}_{\mathcal{O}}\Big(\mathcal{G}_k\big(\hat{\mathcal{F}}_{(c,\hat H)}(Q,f),\bar b_f\big)^{C_f}\Big)$$

where (Q,f) runs over a set of representatives for the set of H-conjugacy classes of selfcentralizing Brauer $(c,\hat H)$-pairs.

Actually, the corresponding form of statement $({}^{\mathcal{K}}\mathrm{Q})$ is nothing but the so-called Equivariant form of Alperin's Conjecture, somewhere stated by Geoffrey Robinson.

11. Finally, we claim that this condition follows from the conditions in [2, §3] and the "compatibility" admitted in [2, Remark 3.1]. Indeed, first of all note that, following the terminology in [2], we are only concerned by the *p-radical* subgroups Q of $\hat{S}$ such that the quotient $\bar{N}_{\hat{S}}(Q) = N_{\hat{S}}(Q)/Q$ admits a block of defect zero or, equivalently, a projective simple module M ; in this case, since the restriction of M to $\bar{C}_{\hat{S}}(Q) = C_{\hat{S}}(Q)/Z(Q)$ remains projective and semisimple, it involves a block $\bar{f}$ of $\bar{C}_{\hat{S}}(Q)$ of defect zero and therefore, denoting by f the corresponding block of $C_{\hat{S}}(Q)$, the Brauer $\hat{S}$-pair (Q, f) is *selfcentralizing* [3, 1.16 and Corollary 7.3]. Then, recalling that there is a bijection between selfcentralizing Brauer pairs and selfcentralizing local pointed groups [3, 7.4], we have in [3, Lemma 15.16] a precise relationship between the sets of selfcentralizing Brauer $\hat{H}$- and $\hat{S}$-pairs.

12. Set $A = H/S$ and, for any irreducible Brauer character θ of $\hat{S}$ in the block d , respectively denote by A_θ and $\hat{H}_\theta$ the stabilizers of θ in A and $\hat{H}$; denoting by c_θ the block of $\hat{H}_\theta$ determined by θ and by $\mathcal{G}_k(\hat{H}_\theta, c_\theta \mid \theta)$ the corresponding direct summand of $\mathcal{G}_k(\hat{H}_\theta, c_\theta)$, it is quite clear that we have

$$\mathrm{rank}_{\mathcal{O}}\big(\mathcal{G}_k(\hat{H}, c)^C\big) = \sum_\theta \mathrm{rank}_{\mathcal{O}}\big(\mathcal{G}_k(\hat{H}_\theta, c_\theta \,|\, \theta)^{C_\theta}\big) \qquad 12.1$$

where θ runs over a set of representatives for the set of H-orbits of irreducible Brauer character of $\hat{S}$ and, for such a θ , C_θ denotes the "stabilizer" of θ in C . Moreover, forgetting the block c_θ and denoting by $\mathcal{G}_k(\hat{H}_\theta \,|\, \theta)$ the corresponding direct summand of $\mathcal{G}_k(\hat{H}_\theta)$, it follows from the so-called *Clifford theory* that we have a canonical isomorphism

$$\mathcal{G}_k(\hat{H}_\theta \,|\, \theta) \cong \mathcal{G}_k(\hat{A}_\theta^{\theta}) \qquad 12.2$$

for a suitable central k^*-extension $\hat{A}_\theta^{\theta}$ of A_θ ; note that, since A_θ is cyclic, the k^*-extension $\hat{A}_\theta^{\theta}$ is split and, since $\mathrm{Aut}(A_\theta)$ is abelian, this automorphism group acts *canonically* on $\hat{A}_\theta^{\theta}$.

13. But, according to the conditions in [2] and to our remark above, θ determines up to S-conjugation a selfcentralizing Brauer $(d, \hat{S})$-pair (Q, f) together with a projective simple $k_*\bar{N}_{\hat{S}}(Q, f)\bar{f}$-module M^* which necessarily has the form

$$M^* \cong W \otimes_k \bar{M}^* \qquad 13.1$$

where W is a projective simple $k_*\bar{C}_{\hat{S}}(Q)\bar{f}$-module, suitable extended to the corresponding central k^*-extension of $\bar{N}_S(Q, f)$, and $\bar{M}^*$ is a projective simple $k_*\hat{\mathcal{F}}_{(d,\hat{S})}(Q, f)$-module, considered as a module over the corresponding central k^*-extension of $\bar{N}_S(Q, f)$; let us respectively denote by θ^* and $\bar{\theta}^*$ the (Brauer) characters of M^* and $\bar{M}^*$.

14. Always according to the conditions in [2], the uniqueness of (Q,f) up to S-conjugation implies that

$$\hat{H}_\theta = \hat{S}{\cdot}N_{\hat{H}_\theta}(Q,f) \quad \text{and} \quad A_\theta \cong N_{\hat{H}_\theta}(Q,f)/N_{\hat{S}}(Q,f) \qquad 14.1.$$

On the other hand, since A_θ is a *cyclic* p'-group, considering the k^*-group $C_{\hat{S}}^{\hat{H}_\theta}(Q,f)$ defined in [3, 15.5.4], it follows from [3, Lemma 15.16] that (Q,f) splits into a set $\{(Q,f_\rho)\}_\rho$ of selfcentralizing Brauer $\hat{H}_\theta$-pairs, where ρ runs over the set of k^*-sections $\mathrm{Hom}_{k^*}(C_{\hat{S}}^{\hat{H}_\theta}(Q,f),k^*)$, and then it is quite clear that $\mathcal{F}_{(d,\hat{S})}(Q,f)$ is a normal subgroup of $\mathcal{F}_{(c_\rho,\hat{H}_\theta)}(Q,f_\rho)$ where c_ρ denotes the block of $\hat{H}_\theta$ determined by (Q,f_ρ) .

15. Consequently, the obvious direct summand $\mathcal{G}_k\big(\hat{\mathcal{F}}_{(c_\rho,\hat{H}_\theta)}(Q,f_\rho)\,|\,\bar{\theta}^*\big)$ of $\mathcal{G}_k\big(\hat{\mathcal{F}}_{(c_\rho,\hat{H}_\theta)}(Q,f_\rho)\big)$ clearly corresponds to blocks of $\hat{\mathcal{F}}_{(c_\rho,\hat{H}_\theta)}(Q,f_\rho)$ of defect zero and, as above, it is not difficult to prove from *Clifford theory* that we have a canonical isomorphism

$$\sum_\rho \mathcal{G}_k\big(\hat{\mathcal{F}}_{(c_\rho,\hat{H}_\theta)}(Q,f_\rho)\,|\,\bar{\theta}^*\big) \cong \mathcal{G}_k(\hat{A}_\theta^{\bar{\theta}^*}) \qquad 15.1$$

where ρ runs over $\mathrm{Hom}_{k^*}(C_{\hat{S}}^{\hat{H}_\theta}(Q,f),k^*)$ and $\hat{A}_\theta^{\bar{\theta}^*}$ is a suitable central k^*-extension of A_θ ; once again, since A_θ is cyclic, the k^*-extension $\hat{A}_\theta^{\bar{\theta}^*}$ is split and, since $\mathrm{Aut}(A_\theta)$ is abelian, this automorphism group acts *canonically* on $\hat{A}_\theta^{\bar{\theta}^*}$.

16. Finally, for any irreducible Brauer character θ of $\hat{S}$ in the block d , from isomorphisms 12.2 and 15.1 we get an isomorphism

$$\mathcal{G}_k(\hat{H}_\theta\,|\,\theta) \cong \sum_\rho \mathcal{G}_k\big(\hat{\mathcal{F}}_{(c_\rho,\hat{H})}(Q,f_\rho)\,|\,\bar{\theta}^*\big) \qquad 16.1$$

where ρ runs over $\mathrm{Hom}_{k^*}(C_{\hat{S}}^{\hat{H}_\theta}(Q,f),k^*)$, which is compatible with the action of $\mathrm{Aut}(A_\theta)$, and therefore with the action of C_θ ; moreover, it is easily checked that it is compatible with the action of the blocks of $\hat{H}$; hence, we get

$$\mathrm{rank}_{\mathcal{O}}\big(\mathcal{G}_k(\hat{H}_\theta,c_\theta\,|\,\theta)^{C_\theta}\big) = \sum_\rho \mathcal{G}_k\big(\hat{\mathcal{F}}_{(c_\rho,\hat{H}_\theta)}(Q,f_\rho)\,|\,\bar{\theta}^*\big) \qquad 16.2$$

where ρ runs over the elements of $\mathrm{Hom}_{k^*}(C_{\hat{S}}^{\hat{H}_\theta}(Q,f),k^*)$ such that (Q,f_ρ) is a Brauer $(c_\theta,\hat{H}_\theta)$-pair. Now, according to equality 12.1, the equality in 10.1 clearly follows from the corresponding sum of these equalities.

17. In conclusion, in order to get Alperin's Conjecture, it suffices to verify that, *for any block* $(c,\hat{H})$ *having a normal sub-block* $(d,\hat{S})$ *of positive defect such that the* k^**-quotient* S *of* $\hat{S}$ *is simple,* H/S *is a cyclic* p'*-group and* $C_H(S)=\{1\}$ *, and for any cyclic subgroup* C *of* $\mathrm{Out}_{k^*}(\hat{H})_c$ *we have*

$$\mathrm{rank}_{\mathcal{O}}\big(\mathcal{G}_k(\hat{H},c)^C\big)=\sum_{(Q,f)}\mathrm{rank}_{\mathcal{O}}\Big(\mathcal{G}_k\big(\hat{\mathcal{F}}_{(c,\hat{H})}(Q,f),\bar{b}_f\big)^{C_f}\Big) \qquad 17.1$$

where (Q,f) *runs over a set of representatives for the set of* H*-conjugacy classes of selfcentralizing Brauer* $(c,\hat{H})$*-pairs and, for such a pair,* $\bar{b}_f$ *denotes the sum of blocks of defect zero of* $\hat{\mathcal{F}}_{(c,\hat{H})}(Q,f)/\mathcal{F}_Q(Q)$ *, and* C_f *the "stabilizer" of* (Q,f) *in* C . We honestly believe that this condition is really easier than the checking demanded in [2]. Of course, this condition could be true whereas the statement (Q) in [4, 1.4] failed!

References

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